Higher-Moment Portfolio Theory

Capitalizing on behavioral anomalies of stock markets.

Yannick Malevergne and Didier Sornette

The capital asset pricing model (CAPM) and the mean-variance portfolio model both rely fundamentally on approximate description of the probability distribution function of asset returns in terms of Gaussian functions, which is necessary in order for the two models to exactly match the expected utility approach (see Von Neuman and Morgenstern [1944]). The mean-variance description is thus at the basis of Markowitz [1959] portfolio theory and of the CAPM (see, for instance, Merton [1990]).

While the variance (or volatility) of portfolio returns provides the simplest way to quantify its fluctuations, it offers only a limited understanding of incurred risks (in terms of fluctuations), because the empirical distributions of returns have fat tails, and the dependencies between assets are only imperfectly accounted for by the covariance matrix. See Lux [1996], Pagan [1996], Gopikrishnan et al. [1998], Litterman and Winkelmann [1998], and Malevergne, Pisarenko, and Sornette [2005].

Accounting for a generalized autoregressive conditional heteroscedastic (GARCH) effect does not really change the picture, because even after correcting returns for volatility clustering, the residual time series still exhibit heavy tails (Cont [2001]). It is thus essential to extend portfolio theory and the CAPM to tackle these empirical facts.

We use general measures of fluctuations that obey the minimum set of consistency properties that we define in Malevergne and Sornette [2005], so that these measures can replace the variance in a theory of portfolio optimization. Absolute central moments and (some) cumulants provide

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extensions of the variance measure of risks, because they satisfy the basic requirement for consistent measures of risks, including in most combinations. The weights can be interpreted in terms of the portfolio manager's aversion to large fluctuations. This approach provides a natural extension of the multimoment investment optimization methodology proposed by Rubinstein [1973], which is based on a linear expansion of the utility function of the economic agent (see Jurczenko and Maillet [2005] for a review of this topic).

We demonstrate how the traditional portfolio optimization framework can be improved when risks are quantified by higher-order centered moments or cumulants. First we show how the concept of efficient frontiers naturally generalizes and how it depends on the chosen risk measure. We then present an analytical understanding of the conditions under which it is possible to simultaneously increase the portfolio return and reduce its large risks quantified by higher-order cumulants, as first reported by Andersen and Sornette [2001]. The multidimensional nature of risks seems to let us break the stalemate of no higher return without more risk, for a class of rational agents who put more weights on large than on small risk.

GENERALIZED EFFICIENT FRONTIERS

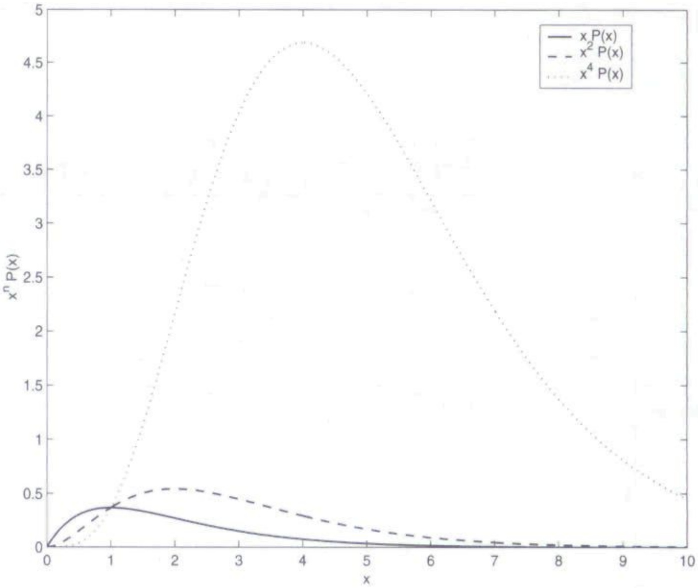
In Malevergne and Sornette [2005] we show centered (absolute) moments and some cumulants to be consistent measures of risks. Here, we will show they can be used to generalize mean-variance Markowitz portfolio theory.

First, let us briefly recall why higher-order moments allow assessment of greater risks. The variance of the return X of an asset, which constitutes the cornerstone of the traditional mean-variance analysis, involves its second moment $\mathrm{E}[X^2]$; more precisely, it is equal to its second centered moment (or moment about the mean) $\mathrm{E}[(X-\mathrm{E}[X])^2]$. Thus, the weight of a given fluctuation of X is proportional to its square.

Due to the decay of the probability density function (pdf) of X for large X bounded from above by $\sim 1/|X|^{1+\alpha}$ with $\alpha > 2$, the largest fluctuations do not contribute significantly to this expectation. To increase their contribution, and thus account for the largest fluctuations, it is natural to invoke higher-order moments of order $n > 2$. The higher n is, the greater the contribution of the rare and large returns in the tail of the pdf.

This phenomenon is demonstrated in Exhibit 1, where we can observe the evolution of the quantity $x^n P(x)$ for $n = 1$, 2, and 4, where $P(x)$ is the density of the standard exponential distribution e^{-x}. The expectation $\mathrm{E}[X^n]$ is then represented geometrically as equal to the area below the curve $x^n P(x)$.

EXHIBIT 1
Function $x^n e^{-x}$ for n = 1, 2, and 4



These curves provide an intuitive illustration of the fact that the main contributions to the moment $\mathrm{E}[X^n]$ of order n come from values of X in the vicinity of the maximum of $x^n P(x)$, which increase quickly with the order n of the moment we consider and the fatter the tail of the pdf of the returns X. For the exponential distribution chosen to construct Exhibit 1, the value of x corresponding to the maximum of $x^n P(x)$ is exactly equal to n. Thus, increasing the order of the moment lets us sample greater fluctuations in asset prices.

The cumulants C_n of the asset return X are defined by the characteristic function of X:

$$\mathrm{E}\left[e^{ikX}\right] = \exp\left(\sum_{n=1}^{+\infty} \frac{(ik)^n}{n!} C_n\right) \tag{1}$$

so that

$$C_1 = \mu \tag{2}$$

$$C_2 = \mu_2 \tag{3}$$

$$C_3 = \mu_3 \tag{4}$$

$$C_4 = \mu_4 - 3\mu_2^{\,2} \tag{5}$$

where μ denotes the expected return of the considered asset, and μ_n represents the centered moment of ordered n.

One can see that the first three cumulants are nothing but the first three centered moments. Thus, using cumulants as a measure of risk that may provide information on risks distinct from centered moments is interesting particularly beyond order three. Cumulants also have a straightforward interpretation: They quantify how far from Gaussian is the distribution of a given risk. Indeed, for a Gaussian distribution, only the two first cumulants are non-zero.

The general optimization problem in generalized portfolio theory is to determine the optimal fraction of wealth $\{w_i\}$ to invest in the asset X_i with expected return $\mu(i)$ such that

$$\begin{cases} \inf_{w_i \in [0,1]} \rho_n(\{w_i\}) \\ \sum_{i \geq 1} w_i = 1 \\ \sum_{i \geq 1} w_i \mu(i) = \mu \end{cases} \tag{6}$$

where ρ_n is a risk measure. It can be an absolute centered moment or a cumulant of order n, for instance. More general examples can be found in Malevergne and Sornette [2005].

Exhibit 2 shows the mean ρ_n efficient frontier resulting from the solution of Equation (6) for a portfolio of 17 assets in the plane $(\mu, \rho_n^{1/n})$, where ρ_n is the risk measure given by the centered moments μ_n of order $n = 2$, 4, 6, and 8. The efficient frontier is concave, as expected from the nature of the optimization problem (6).

The 17 assets are Applied Material, Coca-Cola, EMC, Exxon-Mobil, General Electric, General Motors, Hewlett Packard, IBM, Intel, MCI WorldCom, Medtronic, Merck, Pfizer, Procter & Gamble, SBC Communications, Texas Instruments, and Wal-Mart. The data come from the Center for Research in Security Prices (CRSP) database and cover the period from the end of January 1995 through the end of December 2000, exactly 1,500 trading days. During this period, these companies were among the most highly capitalized on the New York Stock Exchange.

Exhibit 3 describes the main statistical features of the companies in the dataset. Note the high kurtosis of the distributions of returns as well as high values of the observed minimum and maximum returns compared with the standard deviations that clearly underline the non-Gaussian behavior of these assets.

EXHIBIT 2

Generalized Efficient Frontier for Portfolio of 17 Risky Assets

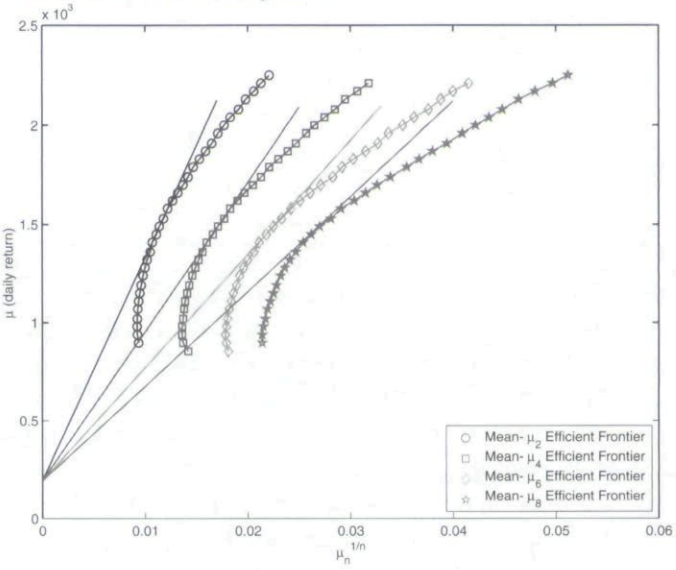


EXHIBIT 3

Descriptive Statistics

	Mean (10^3)	Variance (10^{-3})	Skewness	Kurtosis	min	max
Applied Material	2.11	1.62	0.41	4.68	-14%	21%
Coca-Cola	0.81	0.36	0.13	5.71	-11%	10%
EMC	2.76	1.13	0.23	4.79	-18%	15%
Exxon-Mobil	0.92	0.25	0.30	5.26	-7%	11%
General Electric	1.38	0.30	0.08	4.46	-7%	8%
General Motors	0.64	0.39	0.12	4.35	-11%	8%
Hewlett Packard	1.17	0.81	0.16	6.58	-14%	21%
IBM	1.32	0.54	0.08	8.43	-16%	13%
Intel	1.71	0.85	-0.31	6.88	-22%	14%
MCI WorldCom	0.87	0.85	-0.18	6.88	-20%	13%
Medtronic	1.70	0.55	0.23	5.52	-12%	12%
Merck	1.32	0.35	0.18	5.29	-9%	10%
Pfizer	1.57	0.46	0.01	4.28	-10%	10%
Procter & Gamble	0.90	0.41	-2.57	42.75	-31%	10%
SBC Communication	0.86	0.39	0.06	5.86	-13%	9%
Texas Instruments	2.20	1.23	0.50	5.26	-12%	24%
WalMart	1.35	0.52	0.16	4.79	-10%	9%

For a given value of the expected return μ, the amount of risk measured by $\mu_n^{1/n}$ increases with n, so that there is an additional price to pay for earning more; not only does the μ_2 risk increase, as usual, according to the Markowitz theory, but the high risks also increase faster, the larger n is. That is, the large risks increase more rapidly than the small risks, as the required return increases. This is an important empirical result that has obvious implica-

EXHIBIT 4

Risk Measured by $\mu_n^{1/n}$ for $n = 2, 4, 6, 8$, for Given Value of Expected (daily) Return μ

μ	$\mu_2^{1/2}$	$\mu_4^{1/4}$	$\mu_6^{1/6}$	$\mu_8^{1/8}$
0.10%	0.92%	1.36%	1.79%	2.15%
0.12%	0.96%	1.43%	1.89%	2.28%
0.14%	1.05%	1.56%	2.06%	2.47%
0.16%	1.22%	1.83%	2.42%	2.91%
0.18%	1.47%	2.21%	2.92%	3.55%
0.20%	1.77%	2.65%	3.51%	4.22%

tions for portfolio selection and risk assessment.

For instance, suppose we have an efficient portfolio whose expected (daily) return equals 0.12%, for an annualized return equal to 30%. We can see in Exhibit 4 that the typical risk around the expected return is about twice as high measured by the sixth-order centered moment μ_6 compared to the variance μ_2 and that it is 1.5 times higher measured by the eighth-order centered moment μ_8 compared to the fourth-order centered moment μ_4.

The property we see in Exhibit 2, that the expected return of the portfolios with minimum risk according to the centered moment μ_n drops when n increases, is not the general situation. Exhibits 5 and 6 show the generalized efficient frontiers using the variance C_2 (Markowitz case) and the fourth- and sixth-order cumulants C_4 and C_6 as relevant measures of risks, for two portfolios of two stocks: IBM and Hewlett Packard in Exhibit 5 and IBM and Coca-Cola in Exhibit 6.

Obviously, given a certain amount of risk, the mean return of the portfolio changes when the particular cumulant changes. It is interesting to note that, in Exhibit 5, the minimization of large risks, i.e., with respect to C_6, increases the average return, while in Exhibit 6 the minimization of large risks leads to a drop in the average return. Indeed, in Exhibit 5 one can see that the minimum risk, measured by C_2, C_4, and C_6, is reached for values of the expected daily return approximately equal to 0.08%, 0.10%, and 0.12%, while in Exhibit 6 the minimum risk is reached for values of expected daily return equal to 0.10%, 0.09%, and 0.06%.

This result may be surprising. What does it mean for investor behavior? To solve this puzzle, it is informative to express the cumulants as a function of the centered moments. Such a reparameterization is natural, since the normalized centered moments are increasing functions of their order: $\mu_n^{1/n} \leq \mu_p^{1/p}$ for $n < p$, due to Minkowski inequality. This ordering does not hold for the cumulants, as shown in Exhibits 5 and 6.

Let us consider, for instance, the fourth-order cumulant:

$$C_4 = \mu_4 - 3\mu_2^2 = \mu_4 - 3C_2^2 \qquad (7)$$

An agent assessing the fluctuations of an asset or a portfolio through C_4 shows an aversion for the fluctuations quantified by the fourth central moment μ_4—since C_4 increases with μ_4—but is attracted by the fluctuations measured by the variance—since C_4 declines with μ_2. This behavior is not irrational since it remains globally risk-averse. Indeed, it depicts an agent who tries to avoid the larger risks but is ready to accept the smaller ones.

Note also that, contrary to expansion of the utility function in the spirit of Rubinstein [1973], which necessarily yields linear expressions with respect to the central moments, we have here obtained a non-linear expression. This approach thus lets us characterize a much richer class of attitudes toward risk.

CAN WE HAVE OUR CAKE AND EAT IT TOO?

The properties we have described are characteristic of any agent using the cumulants as risk measures. We can thus understand from a behavioral standpoint why it is possible to have our cake and eat it too in the sense of Andersen and Sornette [2001]. That is, when the cumulants are chosen as risk measures, it may be possible to increase the expected return of a portfolio while reducing its large risks.

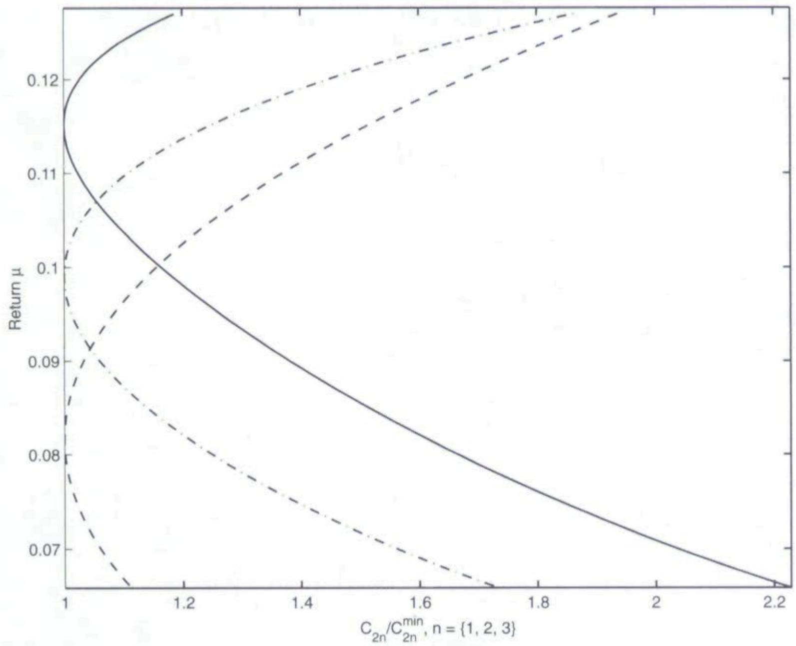
In fact, we can posit a general criterion to determine under what values of the parameters the average return of the portfolio may increase while the large risks drop *at the same time*, thus letting us gain on both accounts (of course, the small risks quantified by the variance will then increase). For two *independent* assets, assuming that the cumulants of order n and $n + k$ of the portfolio admit a minimum for a value of the composition parameter in (0, 1), we can show that the optimal expected return $\mu(n)^*$ that minimizes the risk measured by cumulant C_n is lower than the optimal expected return $\mu(n + k)^*$ that minimizes the risk measured by cumulant C_{n+k}:

$$\mu(n)^* < \mu(n + k)^* \qquad (8)$$

if and only if

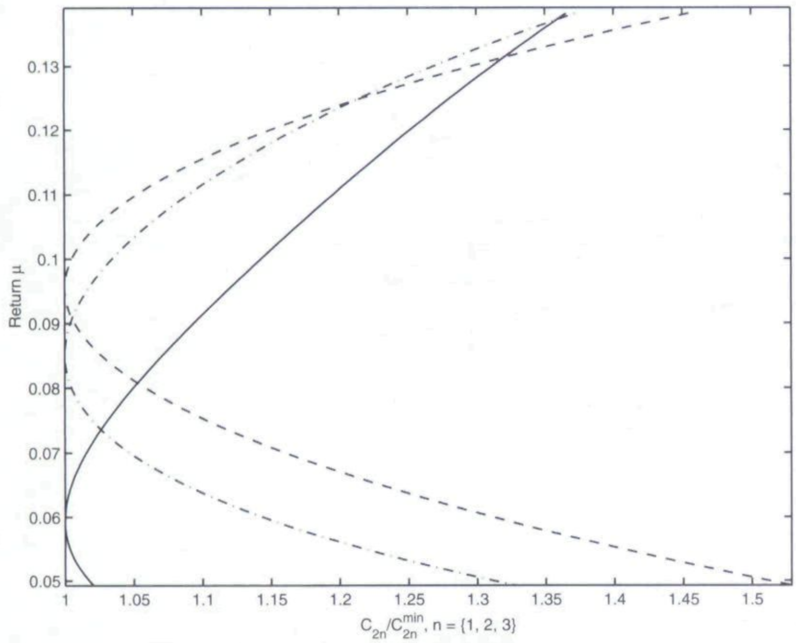
$$([\mu(1) - \mu(2)]) \left[\left(\frac{C_n(1)}{C_n(2)} \right)^{\frac{1}{n-1}} - \left(\frac{C_{n+k}(1)}{C_{n+k}(2)} \right)^{\frac{1}{n+k-1}} \right] > 0 \qquad (9)$$

EXHIBIT 5
Efficient Frontier for Portfolio of IBM and Hewlett-Packard



Dashed line: Efficient frontier with respect to the second cumulant, the standard Markowitz efficient frontier.
Dashed-dotted line: Efficient frontier with respect to the fourth cumulant.
Solid line: Efficient frontier with respect to the sixth cumulant.

EXHIBIT 6
Efficient Frontier for Portfolio of IBM and Coca-Cola



Dashed line: Efficient frontier with respect to the second cumulant, the standard Markowitz efficient frontier.
Dashed-dotted line: Efficient frontier with respect to the fourth cumulant.
Solid line: Efficient frontier with respect to the sixth cumulant.

where $C_n(i)$ is the cumulant of order n for the asset i, with $i = 1, 2$ in this case.

The proof of this result and its generalization to $N > 2$ are given in the appendix. When the tails of the asset distributions remain sufficiently different, this result still holds in the case of dependence between assets.

This last empirical observation in the presence of dependence between assets has not been proved mathematically. It seems, however, reasonable for assets with moderate dependence, while it may fail when the dependence becomes too strong as occurs in periods of turmoil or crisis.

For the assets considered above, we have found $\mu_{IBM} = 0.13$, $\mu_{HWP} = 0.07$, and $\mu_{KO} = 0.05$, and:

$$\frac{C_2(IBM)}{C_2(HWP)} = 1.76 > \left(\frac{C_4(IBM)}{C_4(HWP)}\right)^{\frac{1}{3}} = 1.03 > \left(\frac{C_6(IBM)}{C_6(HWP)}\right)^{\frac{1}{5}} = 0.89 \quad (10)$$

$$\frac{C_2(IBM)}{C_2(KO)} = 0.96 < \left(\frac{C_4(IBM)}{C_4(KO)}\right)^{\frac{1}{3}} = 1.01 < \left(\frac{C_6(IBM)}{C_6(KO)}\right)^{\frac{1}{5}} = 1.06 \quad (11)$$

which shows that, for the portfolio of IBM and Hewlett-Packard, the efficient return is an increasing function of the order of the cumulants while, for the portfolio IBM and Coca-Cola, the inverse occurs. This is exactly what is shown in Exhibits 5 and 6.

The intuition underlying this mechanism is as follows. If a portfolio includes an asset with a rather fat tail (many high risks) but a narrow waist (few low risks) with very little return to gain from it, minimizing the variance C_2 of the return portfolio will overweight this asset, which is mistakenly perceived to have little risk due to its small variance (small waist). In contrast, controlling for the higher risks quantified by C_4 or C_6 prompts reduction in the weight of this asset in the portfolio, and a corresponding increase in the weight of the more profitable assets.

We thus see that the effect of both reducing large risks and increasing profit appears when the asset with the fatter tails, and therefore the narrower central part, has the lower overall return. A mean-variance approach will weight such assets more heavily than would be deemed appropriate from a prudent consideration of large risks and consideration of profits.

From a behavioral point of view, this phenomenon is quite interesting. It can probably be linked to the fact that the main risk measure considered by the agents is the volatility (or the variance), so that the other dimensions of the risk, measured by higher moments, are often neglected. This sometimes offers an opportunity to increase expected return while lowering large risks.

CONCLUSION

Following a consistent set of risk measures adapted to the problem of portfolio risk assessment and optimization, we have described some properties of these generalized optimal portfolios using centered moments and cumulants. We have explained both from a theoretical and a behavioral standpoint that it is possible to have your cake and eat it too. That is, we can increase the expected return of a portfolio while lowering its large risks.

The analytical conditions under which such a phenomenon occurs are of great interest for practical purposes. We can build much more efficient portfolios than those obtained by the standard mean-variance approach, the most common technique used by practitioners.

APPENDIX

Necessary Conditions

We consider N *independent* assets $\{1, \ldots, N\}$, whose expected returns are denoted by $\mu(1), \ldots, \mu(N)$. We aggregate these assets in a portfolio. Let $w_1, \ldots, w_N$ be their weights. Short positions are not allowed, and $\Sigma_i w_i = 1$. The expected return μ of the portfolio is:

$$\mu = \sum_{i=1}^{N} w_i \mu(i) \tag{A-1}$$

Let us assume that the risk of the portfolio is quantified by the cumulants of the distribution of the return of the portfolio. We denote by $\mu(n)^*$ its expected return evaluated for the asset weights that minimize the cumulant of order n.

Assuming that the cumulants $C_n(i)$ have the same sign for all i (according to the condition that a consistent measure of risks must be positive), we will show that the minimum of C_n is obtained for a portfolio whose weights are given by

$$w_i = \frac{\prod_{j\neq i}^{N} C_n(j)^{\frac{1}{n-1}}}{\sum_{j=1}^{N} C_n(j)^{\frac{1}{n-1}}} \tag{A-2}$$

and we have

$$\mu(n)^* = \frac{\sum_{i=1}^{N}\left(\mu(i)\prod_{j\neq i}^{N} C_n(j)^{\frac{1}{n-1}}\right)}{\sum_{j=1}^{N} C_n(j)^{\frac{1}{n-1}}} \tag{A-3}$$

The cumulant of the portfolio is given by

$$C_n = \sum_{i=1}^{N} C_n(i) w_i^n \tag{A-4}$$

subject to the constraint

$$\sum_{i=1}^{N} w_i = 1 \tag{A-5}$$

Introducing a Lagrange multiplier λ, the first-order conditions yield:

$$C_n(i) w_i^{n-1} - \lambda = 0, \ \forall i \in \{1, \cdots, N\} \tag{A-6}$$

so that

$$w_i^{n-1} = \frac{\lambda}{C_n(i)} \tag{A-7}$$

As all the $C_n(i)$ are positive, we can find a λ such that all the w_i are real and positive, which yields the result in Equation (A-2). If we restrict ourself to the case of two assets, we explicitly obtain

$$w^* = \frac{C_n(2)^{\frac{1}{n-1}}}{C_n(1)^{\frac{1}{n-1}} + C_n(2)^{\frac{1}{n-1}}} \tag{A-8}$$

which leads to the expression for μ_n^*:

$$\mu(n)^* = \frac{\mu(1)C_n(2)^{\frac{1}{n-1}} + \mu(2)C_n(1)^{\frac{1}{n-1}}}{C_n(1)^{\frac{1}{n-1}} + C_n(2)^{\frac{1}{n-1}}} \tag{A-9}$$

Thus, after simple algebraic manipulations, we find

$$\mu(n)^* < \mu(n+k)^* \Longleftrightarrow (\mu(1) - \mu(2)) \left(C_n(1)^{\frac{1}{n-1}} C_{n+k}(2)^{\frac{1}{n+k-1}} - C_n(2)^{\frac{1}{n-1}} C_{n+k}(1)^{\frac{1}{n+k-1}} \right) > 0 \qquad \text{(A-10)}$$

which concludes the proof.

When there are more than two assets N, there is no simple condition that ensures $\mu(n)^* < \mu(n+k)^*$. The simplest way to compare $\mu(n)^*$ and $\mu(n+k)^*$ is to calculate these quantities directly using Equation (A-3).

ENDNOTE

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